*RIMS Kôkyûroku Bessatsu*
**B64** (2017), 143–152

# Arithmetic Milnor invariants and multiple power residue symbols in number fields: a précis

By

Masanori MORISHITA*

## Abstract

This is a research announcement of my joint work with Fumiya Amano [AM] concerning a generalization of the Legendre, power residue symbols and the Rédei triple symbol. Such a generalization was firstly discussed by the author at the conference "Algebraic number theory and related topics (2000)" [Mo2]. Afterwards, a significant progress was made by Amano on the 4-tuple quadratic residue symbol over the rationals [A2], and now we are able to introduce $n$-tuple $m$-th power residue symbols for primes of a number field.

## Introduction

Our work is motivated originally by the works of Gauss, about two hundred years ago, on arithmetic of quadratic residue and topology of linking numbers [G1], [G2].

For distinct odd rational primes $p_1$ and $p_2$, the quadratic residue symbol is given by

$$\left(\frac{p_1}{p_2}\right) = \text{Frobenius over } p_2 \text{ in } \mathrm{Gal}(\mathbb{Q}(\sqrt{p_1})/\mathbb{Q}).$$

In view of the analogies between knots and primes [Mo5], it may be regarded as an arithmetic analogue of the mod 2 linking number, $\mathrm{lk}_2(p_1, p_2) \in \mathbb{Z}/2\mathbb{Z}$:

$$\left(\frac{p_1}{p_2}\right) = (-1)^{\mathrm{lk}_2(p_1,p_2)}.$$

In the 19th century, Kummer and Hilbert etc. generalized the quadratic residue symbol to higher power residue symbols in number fields. Let $k$ be a number field

Received March 30, 2015. Revised July 15, 2015.
2010 Mathematics Subject Classification(s): 11R32, 57M05, 57M25.
*Key Words*: *Pro-l Galois groups of link type, Arithmetic Milnor invariants, Multiple power residue symbols, Massey products.*
Partly supported by Grants-in-Aids for Scientific Research (B) 24340005.
*Faculty of Mathematics, Kyushu University, 744, Motooka, Fukuoka 819-0395, Japan.
e-mail: morisita@math.kyushu-u.ac.jp

containing a primitive $m$-th root of unity ($m \geq 2$). For distinct principal prime ideals $\mathfrak{p}_1 = (\pi_1)$ and $\mathfrak{p}_2 = (\pi_2)$, prime to $m$, the $m$-th power reridue symbol is given as

$$\left(\frac{\mathfrak{p}_1}{\mathfrak{p}_2}\right)_m = \text{Frobenius over } \mathfrak{p}_2 \text{ in } \mathrm{Gal}(k(\sqrt[m]{\pi_1})/k).$$

Note that it is natural to assume $\mathfrak{p}_i$ is principal, namely, "null-homologous" in $\mathrm{Spec}(\mathcal{O}_k)$, in order to define the mod $m$ linking number $\mathrm{lk}_m(\mathfrak{p}_1, \mathfrak{p}_2) \in \mathbb{Z}/m\mathbb{Z}$. It may be noteworthy that Hilbert already discussed the power residue symbol for non-principal ideals in § 154 of his Zahlbericht [H].

In 1939, Rédei introduced the triple symbol aiming to generalize the arithmetic of quadratic fields such as Gauss' theory of genera [R]. For distinct certain rational primes $p_1, p_2$ and $p_3$, the Rédei symbol is defined well as

$$[p_1, p_2, p_3]_{\text{Rédei}} = \text{Frobenius over } p_3 \text{ in } \mathrm{Gal}(\mathfrak{R}/\mathbb{Q}),$$

where $\mathfrak{R}$ is determined by $p_1, p_2$ and given concretely by

$$\mathfrak{R} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}),$$

where $\alpha = x + y\sqrt{p_1},\ x^2 - p_1 y^2 - p_2 z^2 = 0$ $(x, y, z \in \mathbb{Z})$. Note that $\mathfrak{R}/\mathbb{Q}$ is a dihedral extension of degree 8, unramified outside $p_1, p_2$ and $\infty$ with ramification index for each $p_i$ being 2. It might not be clear, however, why such a dihedral extension and triple symbol should be considered as a natural generalization of a quadratic field and the Legendre symbol, and it seemed that his work had been overlooked for a long time.

In the late 1990s, Kapranov and the author independently interpreted the Rédei symbol as an arithmetic analogue of a triple linking number for a link [Mi]. For example, the primes 13, 61 and 937 are linked like the Borromean ring [V].

Further the author introduced arithmetic analogues for rational primes of the Milnor invariants (higher order linking numbers) for a link in the 3-sphere [Mo1]-[Mo5]. For example, the mod 2 arithmetic Milnor invariant

$$\mu_2(12\cdots n) \in \mathbb{Z}/2\mathbb{Z}$$

for certain rational primes $p_1, p_2, \ldots, p_n$ describes the decomposition law of $p_n$ in a certain extension $K(n)/\mathbb{Q}$, determined by $p_1, \ldots, p_{n-1}$, which has the following property:

- $K(n)/\mathbb{Q}$ is unramified outside $p_1, \ldots, p_{n-1}$ and $\infty$ with ramification index for each $p_i$ being 2, and the Galois group $\mathrm{Gal}(K(n)/\mathbb{Q}) \simeq N_n(\mathbb{F}_2)$, where $N_n(R)$ stands for the group of $n$ by $n$ upper-triangular unipotent matrices over a commutative ring $R$. In particular, we have

$$(-1)^{\mu_2(12)} = \left(\frac{p_1}{p_2}\right), \quad (-1)^{\mu_2(123)} = [p_1, p_2, p_3]_{\text{Rédei}},$$

and further

$$K(2) = \mathbb{Q}(\sqrt{p_1}), \quad K(3) = \mathfrak{R}$$

for $p_i \equiv 1 \bmod 4$ ([A1], [Mo1]-[Mo4]). This unified interpretation may tell us that Rédei's dihedral extension and triple symbol would be a natural generalization of a quadratic field and the Legendre symbol.
The idea is based on an analogy

$$\begin{array}{ccc} \pi_1^{\text{ét}}(\text{Spec}(\mathbb{Z}) \setminus \{p_1, \ldots, p_n\})(2) & \longleftrightarrow & \pi_1(\mathbb{R}^3 \setminus \{K_1, \ldots, K_n\}) \\ \text{pro-2 Galois group} & & \text{link group,} \end{array}$$

where $\pi_1^{\text{ét}}(\,\cdot\,)(l)$ means the maximal pro-$l$ quotient for a prime number $l$ of the étale fundamental group $\pi_1^{\text{ét}}(\,\cdot\,)$. Using this analogy, we pursue the analogies in arithmetic of the theory of Milnor invariants in link theory.

Since the Milnor invariants are defined for a link in a homology 3-sphere [T], it would be natural to ask the following

**Question**: Can we extend mod 2 arithmetic Milnor invariants to mod $m$ arithmetic Milnor invariants for a finite set $S$ of finite primes in a number field $k$ which contains a primitive $m$-th root of unity and whose class group $H_k = 1$ ?

However, we are faced immediately with the following difficulty: There is an obstruction $B_S$ for the analogy

$$(\star) \qquad \pi_1^{\text{ét}}(\text{Spec}(\mathcal{O}_k) \setminus S)(l) \quad \longleftrightarrow \quad \pi_1(M \setminus \mathcal{L}),$$

where $l$ is a certain prime number and $\mathcal{L}$ is a link in a 3-manifold $M$. We note that the osbtruction $B_S$ is closely related to the group of units $\mathcal{O}_k^{\times}$.

Despite of this difficulty, we have the following

**Result** (rough form): Suppose that $m$ is a power of $l$ and that $k$ contains a primitive $m$-th root of unity $\zeta_m$ and the $l$-class group $H_k(l) = 1$. Then we can introduce mod $m$ arithmetic Milnor invariants $\mu_m(12\cdots n) \in \mathbb{Z}/m\mathbb{Z}$ for a certain set $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of finite primes of $k$ and the $n$-tuple $m$-th power residue symbol in the manner

$$[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m = \zeta_m^{\mu_m(1\ldots n)}.$$

The idea is simple. Namely, we enlarge $S$ so that the obstruction vanishes and we have the analogy $(\star)$, and then we show our invarians are independent of a choice of added auxiliary primes.

Throughout this article, we shall use the following

**Notations**: $l :=$ a fixed prime number, $m :=$ a fixed power of $l$.
$k =$ a finite algebraic number field such that $k$ contains a fixed primitive $m$-th root of unity $\zeta_m$ and the $l$-class group $H_k(l) = 1$.
$S := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} =$ a finite set of finite primes $\mathfrak{p}_i$ of $k$ with $(\mathfrak{p}_i, l) = 1$. Note that $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod m$ $(1 \leq i \leq n)$.
$k_{\mathfrak{p}_i} :=$ the $\mathfrak{p}_i$-adic completion of $k$.
$S_\infty :=$ the set of infinite primes of $k$.
$l_S := \max\{l^e \mid \mathrm{N}\mathfrak{p}_i \equiv 1 \bmod l^e \text{ for } i = 1, \ldots, n\}$.
$k_S(l) :=$ the maximal pro-$l$ Galois extension of $k$, unramified outside $S \cup S_\infty$.
$G_{k,S}(l) := \mathrm{Gal}(k_S(l)/k) = \pi_1^{\text{ét}}(\mathrm{Spec}(\mathcal{O}_k) \setminus S)(l)$.

## § 1. Obstructions and pro-$l$ Galois groups of link type

The *obstruction* $B_S$ is defined by the $\mathbb{F}_l$-vector space

$$B_S := \{a \in k^\times \mid (a) = \mathfrak{a}^l, a \in (k_\mathfrak{p}^\times)^l \text{ for } \mathfrak{p} \in S \cup S_\infty\}/(k^\times)^l.$$

The following theorem is due to Koch.

**Theorem 1.1** ([K, 11.4]). *Assume $B_S = 1$. Then we have*

$$G_{k,S}(l) = \langle x_1, \ldots, x_n \mid x_i^{\mathrm{N}\mathfrak{p}_i - 1}[x_i, y_i] = 1 \ (i = 1, \ldots, n)\rangle,$$

*where $x_i$ is a word representiong a monodromy over $\mathfrak{p}_i$ and $y_i$ is a pro-$l$ word of $x_i$'s representing a Frobenius automorphism over $\mathfrak{p}_i$.*

The obstruction is closely related to the group of units $\mathcal{O}_k^\times$. We let

$$\mathcal{E}_k := \{\mathfrak{p} : \text{ finite prime of } k \mid (\mathfrak{p}, l) = 1, \mathfrak{p} \text{ is inert in } k(\sqrt[l]{\mathcal{O}_k^\times})/k\},$$

which is an infinite set by the Chebotarev density theorem.

**Proposition 1.2** ([AM]). *If $S$ contains a prime in $\mathcal{E}_k$, then $B_S = 1$.*

**Example 1.3.** (1) Let $k = \mathbb{Q}$ and $l = 2$. Then we have $B_S = 1$ for any $S$.
(2) Let $k = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ and $l = 3$. Then $B_S = 1$ if and only if $S$ contains a prime $\mathfrak{p}$ such that $\mathrm{N}\mathfrak{p} \equiv 4$ or $7 \bmod 9$.

## § 2. Arithmetic Milnor invariants and multiple power residue symbols

By Theorem 1.1 and Proposition 1.2, we can enlarge $S$ to $T = S \cup \{\mathfrak{p}_{n+1}, \dots, \mathfrak{p}_t\}$ so that $B_T = 1$ and

$$G_{k,T}(l) = \langle x_1, \dots, x_t \mid x_i^{\mathrm{N}\mathfrak{p}_i - 1}[x_i, y_i] = 1 \ (i = 1, \dots, t) \rangle = F/N,$$

where $F$ is the free pro-$l$ group on the words $x_1, \dots, x_t$ and $N$ is the closed subgroup of $F$ generated normally by $x_i^{\mathrm{N}\mathfrak{p}_i - 1}[x_i, y_i]$ $(i = 1, \dots, t)$. We let

$$\Theta : F \longrightarrow \mathbb{Z}/m\mathbb{Z}\langle\langle X_1, \dots, X_t \rangle\rangle^\times$$

be the mod $m$ Magnus expansion defined by $\Theta(x_i) := 1 + X_i$ for $1 \le i \le t$. We write, for $f \in F$,

$$\Theta(f) = 1 + \sum_{1 \le i_1, \dots, i_r \le t} \mu_m(i_1 \cdots i_r; f) X_{i_1} \cdots X_{i_r}$$

and set

$$\mu_m(i_1 \cdots i_a) := \mu_m(i_1 \cdots i_{a-1}; y_{i_a}) \quad (a > 1),$$

and $\mu_m(i) := 0$ $(1 \le i \le t)$. Define $\Delta_m(1 \cdots n)$ by the ideal of $\mathbb{Z}/m\mathbb{Z}$ generated by $\mu_m(J)$ (multi-indices $J$ running over all cyclic permutations of proper subsequence of $1 \cdots n$) and the binomial coefficients $\binom{l_S}{a}$ $(1 \le a < n)$, and we let

$$\overline{\mu}_m(1 \cdots n) := \mu_m(1 \cdots n) \bmod \Delta_m(1 \cdots n).$$

**Theorem 2.1** ([AM]). *Suppose $2 \le n < l_S$. Then $\overline{\mu}_m(1 \cdots n)$ is an invariant of determined by the $n$-tuple $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ and $m$, more precisely, it is independent of choices of a monodromy and an extension of Frobenius over $\mathfrak{p}_i$ $(1 \le i \le t)$ and is independent of a choice of $T$.*

We call $\overline{\mu}_m(1 \cdots n)$ the mod $m$ *arithmetic Milnor invariant* of the $n$-tuple $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

In the following, we assume for simplicity $\binom{l_S}{a} \equiv 0 \bmod m$ $(1 \le a < n)$ and $\mu_m(j_1 \cdots j_a) = 0$ for any proper subset $\{j_1, \dots, j_a\}$ of $\{1, \dots, n\}$.

We then define the *$n$-tuple $m$-th power residue symbol* by

$$[\mathfrak{p}_1, \dots, \mathfrak{p}_n]_m := \zeta_m^{\mu_m(1 \cdots n)}.$$

We will discuss some properties of our multiple power residue symbols. We define a homomorphism

$$\rho_{m,n} : F \longrightarrow N_n(\mathbb{Z}/m\mathbb{Z})$$

by

$$\rho_{m,n}(f) := \begin{pmatrix} 1 & \mu_m(1;f) & \mu_m(12;f) & \cdots & \mu_m(1\cdots n-1;f) \\ 0 & 1 & \mu_m(2;f) & \cdots & \mu_m(2\cdots n-1;f) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \mu_m(n-1;f) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

It is easily shown that $\rho_{m,n}$ factors through the Galois group $G_{k,T}(l)$.

**Theorem 2.2** ([AM]). *Let $K_m(n)$ be the subfield of $k_T(l)$ fixed by* $\mathrm{Ker}(\rho_{m,n})$. *Then the followings hold.*
(1) *$K_m(n)$ depends only on $S$.*
(2) *$K_m(n)/k$ is unramified outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n-1}$ and $S_\infty$ with ramification index for each $\mathfrak{p}_i$ being $m$, and* $\mathrm{Gal}(K_m(n)/k) \simeq N_n(\mathbb{Z}/m\mathbb{Z})$.
(3) *$\mathfrak{p}_n$ is completely decomposed in $K_m(n)/k$ if and only if $[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m = 1$.*

In view of Theorem 2.2, the following problem is important in the arithmetic of multiple power residue symbols.

**Problem 2.3.** (i) Does the property (2) of Theorem 2.2 characterize $K_m(n)$ uniquely ?
(ii) Can one construct $K_m(n)$ in a concrete manner ?

**Example 2.4.** (1) Let $n = 2$. The classical Kummer and Hibert theory answers Problem 2.3 affirmatively and we have

$$[\mathfrak{p}_1, \mathfrak{p}_2]_m = \left(\frac{\pi_1}{\mathfrak{p}_2}\right)_m \quad \text{for } \mathfrak{p}_1 = (\pi_1).$$

(2) Let $k = \mathbb{Q}$, $l = m = 2$ and $n = 3$. Amano [A1] answers Problem 2.3 affirmatively so that $K_2(3)$ coincides with Rédei's dihedral extension $\mathfrak{R}$, and we have

$$[p_1, p_2, p_3]_2 = [p_1, p_2, p_3]_{\text{Rédei}}.$$

(3) (Triple cubic residue symbol) This is a new case where $k = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$, $l = m = 3$ and $n = 3$. Let $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ with $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod 9$. Then there is $\pi_i \in \mathbb{Z}[\zeta_3]$ uniquely such that $\mathfrak{p}_i = (\pi_i)$ and $\pi_i \equiv 1 \bmod (1-\zeta_3)^3$. Assume that

$$\left(\frac{\pi_i}{\pi_j}\right)_3 = 1 \ \ (i \neq j).$$

By Theorem 2.2, the extension $K_3(3)/k$ is a mod 3 Heisenberg extension of degree 27, unramified outside $\mathfrak{p}_1, \mathfrak{p}_2$ with ramification index for each prime $\mathfrak{p}_i$ being 3. Then we have the following theorem.

**Theorem 2.5** ([AM]). (1) *The property* (2) *of Theorem* 2.2 *with* $m = n = 3$ *characterizes* $K_3(3)$ *uniquely and we have*

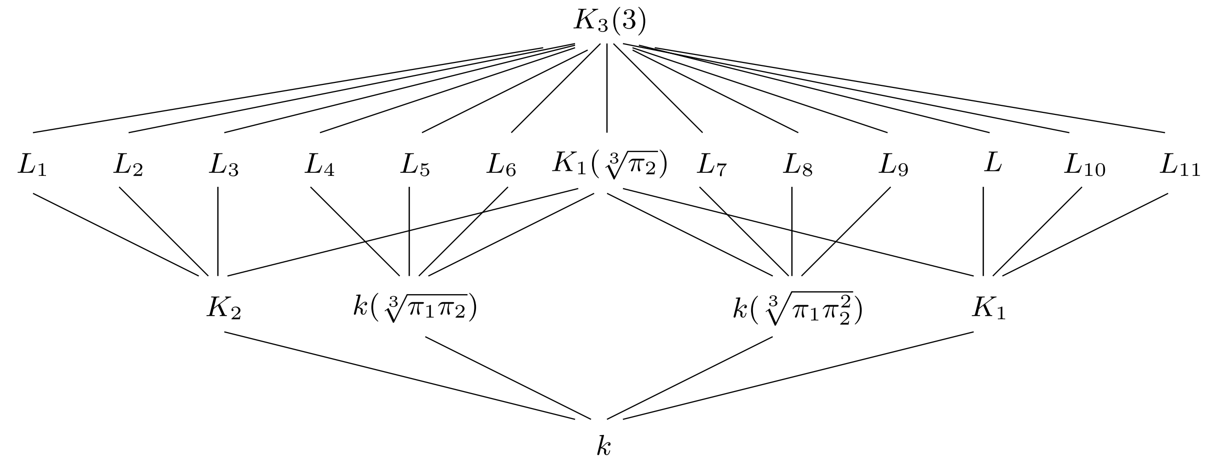
$$K_3(3) = k(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}),$$

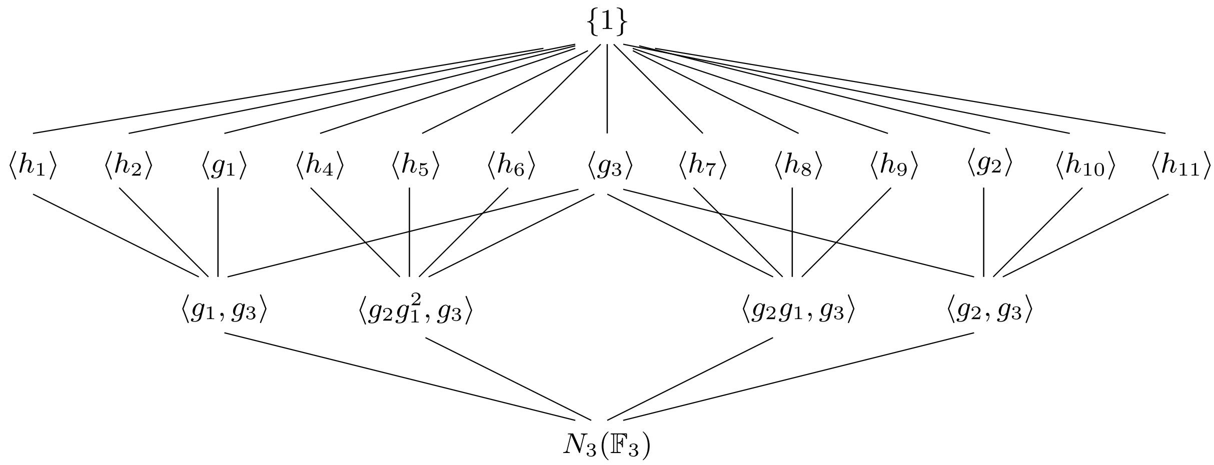
*where* $\theta = x + y\sqrt[3]{\pi_1} + z(\sqrt[3]{\pi_1})^2$ *is an algebraic integer in* $k(\sqrt[3]{\pi_1})$ *satisfying* $x^3 + \pi_1 y^3 + \pi_1^2 z^3 - 3\pi_1 xyz - \pi_2^3 w^2 = 0$ $(x, y, z, w \in \mathbb{Z}[\zeta_3])$. (For more detailed requirements of $\theta$, refer to [AM; Section 4])
(2) *We have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \frac{\mathrm{Frob}_{\mathfrak{p}_3}(\sqrt[3]{\theta})}{\sqrt[3]{\theta}},$$

*where* $\mathrm{Frob}_{\mathfrak{p}_3}$ *is an extension of Frobenius automorphism over* $\mathfrak{p}_3$ *in* $K_3(3)/k$.

All subgroups of $\mathrm{Gal}(K_3(3)/k)$ and the corresponding intermidiate fields are illustrated as follows.



$$N_3(\mathbb{F}_3) = \left\langle g_1, g_2, g_3 \,\middle|\, \begin{array}{l} g_1^3 = g_2^3 = g_3^3 = 1 \\ g_2g_1 = g_3g_2g_1,\ g_3g_1 = g_1g_3,\ g_3g_2 = g_2g_3 \end{array} \right\rangle,$$

$$g_1 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 0\,0\,1 \end{pmatrix},\quad g_2 = \begin{pmatrix} 1\,1\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix},\quad g_3 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} (= [g_2, g_1]).$$

$$\begin{array}{l} g_1 : (\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta_1}, \sqrt[3]{\theta_2}, \sqrt[3]{\theta_3}) \mapsto (\zeta_3 \sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta_2}, \sqrt[3]{\theta_3}, \sqrt[3]{\theta_1}), \\ g_2 : (\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta_1}, \sqrt[3]{\theta_2}, \sqrt[3]{\theta_3}) \mapsto (\sqrt[3]{\pi_1}, \zeta_3 \sqrt[3]{\pi_2}, \sqrt[3]{\theta_1}, \zeta_3^2 \sqrt[3]{\theta_2}, \zeta_3 \sqrt[3]{\theta_3}), \\ g_3 : (\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta_1}, \sqrt[3]{\theta_2}, \sqrt[3]{\theta_3}) \mapsto (\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \zeta_3^2 \sqrt[3]{\theta_1}, \zeta_3^2 \sqrt[3]{\theta_2}, \zeta_3^2 \sqrt[3]{\theta_3}), \end{array}$$

where $\theta_1 := \theta, \theta_2 := g_1(\theta), \theta_2 := g_1^2(\theta)$.

$$\begin{array}{c} h_1 = g_1g_3,\ \ h_2 = g_1g_3^2,\ \ h_4 = g_2g_1^2,\ \ h_5 = g_2g_1^2g_3,\ \ h_6 = g_2g_1^2g_3^2, \\ h_7 = g_2g_1,\ \ h_8 = g_2g_1g_3,\ \ h_9 = g_2g_1g_3^2,\ \ h_{10} = g_2g_3,\ \ h_{11} = g_2g_3^2. \end{array}$$

## § 3. Massey products

It is known that Milnor invariants of a link are interpreted as Massey products in the cohomology of the link complement [T]. Similarly, we can interpret our multiple power residue symbols as Massy products in the étale cohomology of the complement $\mathfrak{X} := \mathrm{Spec}(\mathcal{O}_k) \setminus T$, where $T$ is, as in §2, a finite set containing $S$ such that $B_T = 1$. This is a generalization of the cup product interpretation of the power residue symbol [S]. We keep the same notations as in Section 2.

In order to define the Massey product structure in étale cohomology, we use Verdier's construction of étale cohomology presented in [AGV; Exposé V]. Let $U_\bullet$ be a hypercovering on $\mathfrak{X}_{\text{ét}}$ and let $C^j := C^j(U_\bullet, \mathbb{Z}/m\mathbb{Z})$ be the Čech $j$-cochains ($j \geq 0$) associated to $U_\bullet$ with coefficients in the constant sheaf $\mathbb{Z}/m\mathbb{Z}$. Consider the differential graded algebra $(C^\bullet := \oplus_{j\geq 0} C^j, d^\bullet)$ equipped with multiplication given by the Alexander-Whiteney cup product $\cup$. Then, by the general procedure to define Massey products [Ma], we have the Massey product structure on the étale cohomology $H^\bullet(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}) = \varinjlim H^*(C^\bullet(U_\bullet, \mathbb{Z}/m\mathbb{Z}))$, where the limit is taken over the homotopy category of hypercoverings $U_\bullet$.

Let us explain concretely the triple Massey product concerning Example 2.4 (3). For the general case, we refer to [AM; Section 5]. Let $\chi_i \in H^1(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ be the Kronecker dual to the monodromies $x_j \in G_{k,T}(3)$, namely, $\chi_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq \#T$. By the assumption $\left(\dfrac{\pi_i}{\pi_j}\right)_3 = 1$ $(i \neq j)$, we have $\chi_{12}, \chi_{23} \in C^1$ such that

$$\chi_1 \cup \chi_2 = d\chi_{12}, \quad \chi_2 \cup \chi_3 = d\chi_{23}.$$

Then the triple Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle \in H^2(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ is defined by the cohomology class of the 2-cocycle

$$\chi_1 \cup \chi_{23} + \chi_{12} \cup \chi_3.$$

On the other hand, let $\delta_3 \in H_2(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ be the image of the canonical generator of $H_2(\mathrm{Spec}(k_{\mathfrak{p}_3})_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$, which represents a "boundary of a tubular neighborhood" of $\mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}_3)$, under the natural homomorphism $H_2(\mathrm{Spec}(k_{\mathfrak{p}_3})_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}) \to H_2(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$.

**Theorem 3.1** ([AM]). $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \zeta_3^{-\langle \chi_1, \chi_2, \chi_3 \rangle(\delta_3)}$.

Finally we present an important problem left unsolved:

**Problem** (reciprocty law). Study the behavior of $[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m$ under permutations of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

## References

[A1] Amano, F., On Rédei's dihedral extension and triple reciprocity law, Proc. Japan Acad., **90**, Ser. A (2014), 1–5.

[A2] Amano, F., On a certain nilpotent extension over $\mathbb{Q}$ of degree 64 and the 4-th multiple residue symbol, Tohoku Math. J. **66** No.4 (2014), 501 –522.

[A3] Amano, F., Arithmetic of nilpotent extensions and multiple residue symbols, Thesis, Kyushu University, 2014.

[AGV] Artin, M., Grothendieck, A. and Verdier, J.-L., Théorie des topos et cohomologie étale des schémas, SGA4 Tome 2, Lecture Notes in Mathematics, vol. **270** Springer, Berlin-New York 1972.

[AM] Amano, F., Morishita, M., Arithmetic Milnor invariants and multiple power residue symbols in number fields, arXiv:1412.6894.

[G1] Gauss, C. F., Disquisitiones arithmeticae, Translated into English by A. Arthur, S. J. Clarke, Yale University Press, New Haven, Conn.-London 1966.

[G2] Gauss, C. F., Zur mathematischen Theorie der electrodynamischen Wirkungen, Werke **V**, (1833), p 605.

[H] Hilbert, D., Die Theorie der algebraischen Zahlkörper, Jahresbericht der Deutschen Abhandlungen. Band I: Zahlentheorie. Zweite Auflage Springer, 1970.

[K] Koch, H., Galoissche Theorie der $p$-Erweiterungen, Springer-Verlag, VEB Deutscher Verlag der Wissenschaften, Berlin, 1970.

[Ma] May, J. P., Matric Massey products, J. Algebra **12** (1969), 533–568.

[Mi] Milnor, J., Isotopy of links, in Algebraic Geometry and Topology, A symposium in honor of S. Lefschetz ( edited by R.H. Fox, D.C. Spencer and A.W. Tucker), 280–306 Princeton University Press, Princeton, N.J., 1957.

[Mo1] Morishita, M., Milnor's link invariants attached to certain Galois groups over **Q**, Proc. Japan Acad. Ser, A **76** (2000), 18–21.

[Mo2] Morishita, M., Knots and primes, 3-manifolds and number fields (Japanese), Algebraic number theory and related topics (2000), RIMS Kôkyûroku **200** (2001), 103 – 115.

[Mo3] Morishita, M., On certain analogies between knots and primes, J. Reine Angew. Math. **550** (2002), 141–167.

[Mo4] Morishita, M., Milnor invariants and Massey products for prime numbers, Compos. Math., **140** (2004), 69–83.

[Mo5] Morishita, M., Knots and Primes - An introduction to arithmetic topology, Universitext, Springer, London, 2012.

[R] Rédei, L., Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper I, J. Reine Angew. Math., **180** (1939), 1-43.

[S] Serre, J.-P., Corps locaux, Publications de l'Université de Nancago, Hermann, Paris, 1968.

[T] Turaev, V., The Milnor invariants and Massey products, (Russian) Studies in topology, II. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **66**, (1976), 189–203, 209–210.

[V] Vogel, D., On the Galois group of 2-extensions with restricted ramification, J. Reine Angew. Math., **581** (2005), 117–150.